Title	Stability in Linear Systems of Difference and Differential-Difference Equations (函数微分方程式の解の定性的研究)
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Citation	数理解析研究所講究録 (1977), 288: 50-70
Issue Date	1977-02
URL	http://hdl.handle.net/2433/106140
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
288巻 1977年 50-70

Stability in linear systems of difference and differential-difference equations

by

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Summary: The purpose of this paper is to discuss the present status of the theory of stability for autonomous linear systems of difference and differential-difference equations. In particular, we will be concerned with the preservation of stability under perturbations in the delays.

§1. Introduction and basic properties. For any real number $r \geqslant 0$, let $C = C([-r, 0], R^n)$ be the space of continuous functions from $[-r, 0]$ into R^n with the topology of uniform convergence. Suppose $D, L : C \longrightarrow R^n$ are continuous linear operators with

$$(1.1) \qquad D\phi = \phi(0) - \int_{-r}^{0} [d\mu(\theta)]\phi(\theta), \quad L\phi = \int_{-r}^{0} [d\eta(\theta)]\phi(\theta)$$

where μ, η are $n \times n$ matrix functions of bounded variation with

$$(1.2) \qquad \mathrm{Var}_{[-s,0]}\mu \longrightarrow 0 \quad \text{as} \quad s \longrightarrow 0.$$

We may always assume μ, η defined on $(-\infty, \infty)$ by defining $\mu(\theta) = \mu(0)$, $\eta(\theta) = \eta(0)$, $\theta \in [0, \infty)$, $\mu(\theta) = \mu(-r)$, $\eta(\theta) = \eta(-r)$,

$\theta \in (-\infty, -r]$.

An autonomous linear neutral functional differential equation NFDE(D, L) is a relation

$$\frac{d}{dt} Dx_t = Lx_t \tag{1.3}$$

where $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$.

By a solution of (1.3) through $\phi \in C$, we mean a function $x : [-r, \infty) \longrightarrow R^n$ which is continuous, $x_0 = \phi$, and Dx_t is continuously differentiable and satisfies (1.3) on $[0, \infty)$. The proof of the existence of a solution through any $\phi \in C$ may be found in [1]. If $x(\phi)$ designates the solution through ϕ and $T_{D,L}(t) : C \longrightarrow C$, $t \geq 0$, is defined by

$$T_{D,L}(t)\phi = x_t(\phi), \; t \geq 0, \; \phi \in C, \tag{1.4}$$

then $T_{D,L}(t)$, $t \geq 0$, is a strongly continuous semigroup of bounded linear operators with infinitesimal generator $A_{D,L}$ and domain $D(A_{D,L})$ given by

$$\begin{aligned} D(A_{D,L}) &= \{\phi \in C : \dot{\phi} \in C, \; D\dot{\phi} = L\phi\} \\ A_{D,L}\phi &= \dot{\phi} \end{aligned} \tag{1.5}$$

If $\sigma(B)$, $P\sigma(B)$ denote respectively the spectrum, point spectrum of a linear operator B, then one also can prove that (see [1])

$$(1.6)\quad \sigma(A_{D,L}) = P\sigma(A_{D,L}) = \{\lambda : \det\Delta(\lambda) = 0,\ \Delta(\lambda) = \lambda D(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I)\}$$

The equation $\det\Delta(\lambda) = 0$ is called the characteristic equation.

§2. Exponential bounds. With the notation of Section 1, it is easy to ask interesting nontrivial questions.

Problem 2. 1. What is the relationship between $\sigma(T_{D,L}(t))$ and $\exp\sigma(A_{D,L})$?

This problem is difficult and an easier problem can be stated as follows. The order $a_{D,L}$ of the seimgroup $T_{D,L}(t)$ is defined as

$$(2.1)\qquad a_{D,L} = \inf\{a \in R : \text{there is a } K = K(a) \text{ such that } \|T_{D,L}(t)\| \leqslant K \exp at,\ t \geqslant 0\}.$$

Problem 2. 2. Is $a_{D,L} = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A_{D,L})\}$?

At this time,neither of these problems is solved for general D, L as above. For the case in which the function η has no singular part, partial results are known. Since we are concerned with stability, we concentrate on Problem 2. 2. Suppose D satisfies (1.2) and

$$D = D_0 + D_1$$

(2.2) $$D_0\phi = \phi(0) - \sum_{k=1}^{\infty} A_k\phi(-r_k)$$

$$D_1\phi = \int_{-r}^{0} A(\theta)\phi(\theta)d\theta$$

where the n×n matrix A is Lebesgue integrable and the n×n matrices A_k satisfy $\sum_{k=1}^{\infty}|A_k| < \infty$ and $0 \leqslant r_k \leqslant r$.

Theorem 2.1([2]) $a_{D,L} = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$.

We give an outline of a proof of this theorem since it will illustrate many of the techniques used in neutral equations as well as motivate other problems.

The first step is to analyze in detail the difference equation

(2.3) $$D_0 y_t = 0, \; y_0 = \psi \in C_{D_0}$$

where C_{D_0} is defined by

(2.4) $$C_{D_0} = \{\psi \in C : D_0\psi = 0\}$$

Equation (2.4) generates a strongly continuous semigroup of linear transformations $T_{D_0}(t) : C_{D_0} \longrightarrow C_{D_0}$, $t \geqslant 0$, with

$T_{D_0}(t)\psi = y_t(\psi)$ for $\psi \in C_{D_0}$. The infinitesimal generator A_{D_0} and its domain $D(A_{D_0})$ are given by

(2.5) $$A_{D_0}\psi = \dot{\psi},\ D(A_{D_0}) = \{\psi \in C_{D_0} : \dot{\psi} \in C_{D_0}\}$$

also,

(2.6) $$\sigma(A_{D_0}) = P\sigma(A_{D_0}) = \{\lambda : \det\Delta_{D_0}(\lambda) = 0,\ \Delta_{D_0}(\lambda) = I - D_0 e^{\lambda \cdot} I\}$$

<u>Lemma 2. 1.</u> The order a_{D_0} of $T_{D_0}(t)$ is given by

$$a_{D_0} = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A_{D_0})\} \overset{\text{def}}{=} \beta$$

<u>Idea of the proof</u>: By using Laplace transform, one shows that the solution $y(\psi)(t)$ of (2.3) can be represented as

$$y(t) = -\sum_{k=1}^{\infty} \int_{-r}^{0-} [d_\beta A_k Y(t-\beta-r_k)]y(\beta),$$

where $Y(t)$ is the fundamental matrix solution; that is, $Y(t)$ is the n×n matrix solution of the equation

$$Y(t) = \begin{cases} 0, & t < 0 \\ I + \sum_{k=1}^{\infty} A_k Y(t - r_k), & t \geq 0, \end{cases}$$

which is of bounded variation and continuous from the right. The Laplace transform of $Y(t)$ is $\lambda^{-1}\Delta_{D_0}^{-1}(\lambda)$. Since $a_{D_0} \geq \beta$, to prove the lemma, it remains to show that, for any $a > \beta$, there is a constant $k = k(a)$ such that

$$|Y(t)| \leq ke^{at}, \quad \operatorname{var}_{[t-r,t]} Y \leq ke^{at}, \quad t \geq 0.$$

To obtain this estimate using the inverse Laplace transform, one makes use of the fact that $[\det\Delta_{D_0}(\lambda)]^{-1}$ is an analytic almost periodic function on each line $\operatorname{Re}\lambda = \alpha$ for which $\det\Delta_{D_0}(\lambda)$ is bounded away from zero. Therefore, there exist real γ_k such that

$$[\det\Delta_{D_0}(\lambda)]^{-1} = \sum_{k=1}^{\infty} b_k e^{\lambda\gamma_k}, \quad \sum_{k=1}^{\infty} |b_k| e^{\alpha\gamma_k} < \infty.$$

One now uses these relations to estimate $Y(t)$.

The next step is to analyze the nonhomogeneous linear equation

$$D_0 y_t = h(t), \quad h \in C([0,\infty), R^n). \tag{2.7}$$

The argument used in proving the next lemma is valid for general D_0 satisfying (1.1),(1.2) and can even be adapted to the case where D_0 depends on t. For simplicity, we consider only D_0 independent of t. The ideas follow [3].

Let $\Phi_0 = (\phi_1^0, \ldots, \phi_n^0)$, $\phi_i^0 \in C$, be n functions such that $D_0\Phi_0 = I$ and let $\Psi_0 = I - \Phi_0 D_0$. Then $\Psi_0 : C \longrightarrow C_{D_0}$ is a continuous projection.

If $y(\psi, h)$ is the solution of (2.7) through ψ, then

$$y(\psi, h) = y(\Psi_0\psi, 0) + y(\Phi_0 h(0), h).$$

If we define $K_{D_0}(t) : C([0, t], R^n) \longrightarrow R^n$ by $K_{D_0}(t) = y(\Phi_0 h(0), h)(t)$, then $K_{D_0}(t)$ is a continuous linear operator and one can prove

Lemma 2. 2. For any $a > a_{D_0}$, $a \neq 0$, there is a constant $k = k(a)$ such that

$$\|K_{D_0}(t)\| \leqslant \begin{cases} ke^{at} & \text{if } a > 0 \\ k & \text{if } a < 0 \end{cases}$$

Idea of the proof: Suppose $a = a_{D_0} + 2\varepsilon$, $y(t) = z(t)e^{at}$, $H(t) = e^{-at}h(t)$, $\tilde{D}_0\phi = D_0 e^{a\cdot}\phi$. Then

$$\tilde{D}_0 z_t = H(t), \ |T_{\tilde{D}_0}(t)| \leqslant ke^{-\varepsilon t}, \ t \geqslant 0.$$

One now uses fairly standard estimates over intervals $[j\sigma, j\sigma + \sigma]$, $j = o, 1, 2, \ldots$ for an appropriate σ.

Lemma 2. 3. $T_{D,L}(t) = T_{D_0}(t)\Psi_0 + U(t)$, $t \geqslant 0$ where $U(t)$ is completely continuous.

Idea of the Proof: The function $U(t)\phi$ has initial data $U(0)\phi = \Phi_0 D_0\phi$. Therefore, as ϕ ranges over C, $U(0)\phi$ ranges over a finite dimensional space. Furthermore,

$$D_0[U(t+\tau)\phi - U(t)\phi] = D_1(T_{D,L}(t+\tau)\phi - T_{D,L}(t)\phi) + \int_t^{t+\tau} L(T_{D,L}(s)\phi)ds$$

Now use Lemma 2.2 and the above remark about the initial data.

Completion of the proof of Theorem 2. 1. Let $r_e(A)$, $\gamma(A)$ be, respectively, the radius of the essential spectrum and the radius of the spectrum of a bounded operator A. Lemma 2.3 implies $r_e(T_{D,L}(r)) = r_e(T_{D_0}(r)\Psi_0)$. Also, $(T_{D_0}(r)\Psi_0)^k = T_{D_0}(kr)\Psi_0$ implies $\gamma(T_{D_0}(r)\Psi_0) = \gamma(T_{D_0}(r))$. The next step is to prove that $\gamma(T_{D_0}(r)) = r_e(T_{D_0}(r))$. This uses the almost periodicity of $\det\Delta_{D_0}(\lambda)$ on a line $\mathrm{Re}\lambda$ = constant.

The proof of the theorem is now completed by observing that there exist roots λ_j of $\det\Delta(\lambda) = 0$ of large modulus such that $\mathrm{Re}\lambda_j \longrightarrow a_{D_0}$ as $j \longrightarrow \infty$.

This latter argument also shows that

$$a_{D_0} \leqslant a_{D,L} \tag{2.8}$$

In particular, if the zero solution of the NFDE(D, L) is uniformly asymptotically stable (u.a.s.), then the zero solution of the difference equation $D_0 y_t = 0$ is u.a.s.

§3. Preservation of u.a.s. We now turn to the preservation of u.a.s. when perturbations are made in the parameters in the equation. The next result can be proved either directly from the characteristic equation or from the variation of constants formula for the equation

$$\frac{d}{dt}[Dx_t - H(t)] = Lx_t + h(t).$$

Theorem 3. 1. If the NFDE(D, L) is u.a.s., then there is a neighborhood V of (D, L) in the operator topology such that the NFDE($\bar{D}$, $\bar{L}$) is u.a.s. for all $(\bar{D}, \bar{L}) \in V$.

Theorem 3. 1 says that we can make small variations in the coefficients of D, L and not destroy the property of u.a.s. For the case when $D = D_0 + D_1$ in (2.2), this means we can change the norm of the matrices A_k and the L^1 norm of $A(\theta)$ a small amount and preserve u.a.s. We now study the effect of variations in the delays. To do this, we suppose

$$\begin{aligned}
&D(r) = D_0(r) + D_1\\
&L(s) = L_0(s) + L_1\\
&D_0(r)\phi = \sum_{k=1}^{N} A_k\phi(-r_k),\ r = (r_1, \ldots, r_N)\\
(3.1)\qquad &L_0(s)\phi = \sum_{k=1}^{M} B_k\phi(-s_k),\ s = (s_1, \ldots, s_M)\\
&D_1\phi = \int_{-r_0}^{0} A(\theta)\phi(\theta)d\theta\\
&L_1\phi = \int_{-s_0}^{0} B(\theta)\phi(\theta)d\theta
\end{aligned}$$

where each $r_k > 0$, $s_k > 0$ and A, B are L^1 matrix functions.

Definition 3. 1. We say the NFDE(D(r), L(s)) is stable locally *in the delays* (s.l.d.) at (r, s) if there exist intervals $I_r \subset (R^+)^N$, $I_s \subset (R^+)^M$ of r, s such that the NFDE$(D(\bar{r}), L(\bar{s}))$ is u.a.s. for each $(\bar{r}, \bar{s}) \in I_r \times I_s$. We say *$D(r)$ is s.l.d. at r* if the there is an interval $I_r \subset (R^+)^N$ of r such that the zero solution of $D_0(\bar{r})y_t = 0$ is u.a.s for all $\bar{r} \in I_r$.

From Inequality (2.8), we have the following

Lemma 3. 1. If the NFDE$(D(r), L(s))$ is s.l.d. at (r, s), then $D_0(r)$ is s.l.d. at r.

From this lemma, it is clear that we must understand when $D_0(r)$ is s.l.d. ar r. This is the topic discussed in the next section.

§4. Preservation of stability for $D_0(r)$. Suppose $D(r)$ is defined as in Relation (3.1). The behavior of the solutions of $D_0(r)y_t = 0$ as a function of r is very complicated. The following simple example from [4] shows that u.a.s. may not be preserved when one makes small changes in r. Suppose

$$D_0 y_t = y(t) + \frac{1}{2}y(t-1) + \frac{1}{2}y(t-2)$$

The characterisctic equation is

$$1 + \frac{1}{2}e^{-\lambda} + \frac{1}{2}e^{-2\lambda} = 0$$

and $|e^{\lambda}| = 1/2$, which implies $\mathrm{Re}\lambda = -\ln 2 < 0$. For any integer n, the equation

$$y(t) + \frac{1}{2}y(t-1-\frac{1}{2n+3}) + \frac{1}{2}y(t-2) = 0$$

has the solution $y(t) = \sin(n+\frac{3}{2})\pi t$. Thus, there is a $\lambda \in \sigma(A)$ with $\mathrm{Re}\lambda = 0$ and u.a.s is not preserved even though n may be very large.

The following result which will not be proved is the best one available at this time.

<u>Theorem 4. 1</u>[4]. The following statements are equivalent

(i) $D_0(r)$ is s.l.d. at one $r \in (R^+)^N$

(ii) $D_0(r)$ is u.a.s. for all $r \in (R^+)^N$

(iii) $D_0(r)$ is u.a.s. for one fixed $r \in (R^+)^N$ with the components r_k rationally independent.

(iv) If $\gamma(B)$ is the spectral radius of a matrix B, then $\gamma_0(A) \overset{\text{def}}{=} \sup\{\gamma(\sum_{k=1}^{N} A_k e^{i\theta_k}) : \theta_k \in [0, 2\pi], k=1,2,\ldots,N\}<1$

For the scalar difference equation, $D_0(r)y_t = 0$, the number $\gamma_0(A)$ in (iv) is given by

$$\gamma_0(A) = \sum_{k=1}^{N} |A_k|$$

Thus, for the scalar equation, $D_0(r)$ is s.l.d. if and only if $\sum_{k=1}^{N} |A_k| < 1$. This result was first proved in [5]. All of the statements in Theorem 4.1 concern the nature of the roots of a characteristic equation

$$1 - \sum_{k=1}^{p} \alpha_k e^{-\lambda\omega_k} = 0$$

where the α_k are real scalars and each $\omega_k > 0$ is a linear function of the components of r. If the ω_k are allowed to vary independently, this corresponds to the case of a sclalr $D_0(r)$ discussed in [5]. For the matrix case, the ω_k do not vary independently when the r_k vary independently. For this latter case, the equivalence of (i) and (ii) was first proved in [6]. To apply Theorem 4.1 to a scalar difference equation with delays which are not independent, one must transform the scalar equation to a system which preserves the characteristic equation. When applied to the equation

$$(4.1) \qquad y(t) - ay(t-r_1) - by(t-r_2) - cy(t-r_1-r_2) = 0$$

one has s.l.d.at (r_1, r_2) if and only if

$$1 + a > |b+c|, \quad 1 - a > |b-c|$$

which is larger than the set $1 > |a| + |b| + |c|$ corresponding to letting the three delays vary independently.

Using [7, Theorem 7, p. 70] and Theorem 3.2, one can also prove the following interesting result.

Theorem 4. 3. The following statements are equivalent:

(i) $D_0(r)$ is s.l.d. at r

(ii) There exist constants $k > 0$, $\alpha > 0$ such that, for every $r \in (R^+)^N$, $\|T_{D_0(r)}(t)\| \leqslant ke^{-\alpha t}$, $t \geqslant 0$

(iii) There exist an $\alpha > 0$ such that $\sup\{a_{D_0(r)} : r \in (R^+)^N\} < -\alpha$.

From Theorem 4.3, there is an equivalent norm in C such that $\|T_{D_0(r)}(t)\| \leqslant e^{-\alpha t}$, $t \geqslant 0$. Therefore, if $g_0(r) : C \longrightarrow C$ is defined by $g_0(r)\phi = \phi(0) - D_0(r)\phi$, then, in this norm, $\|g_0(r)\| \leqslant 1$. From Theorem 4.1 and computations on Equation(4.1) is seems that $\|g_0(r)\| < 1$ and we formulate this as a conjecture.

Conjecture. $D_0(r)$ is s.l.d. at r if and only if there exists an equivalent norm in C such that $\|D_0(r)\phi - \phi(0)| < |\phi|$ for all $r \in (R^+)^N$ and all $\phi \in C$. Another interesting problem is the following one.

Problem: If $D_0(r)$ is s.l.d., for what continuous functions $r(t) \in (R^+)^N$ is $D_0(r(t))$ u.a.s.?

§5. Preservation of stability for NFDE $(D(r), L(s))$. Suppose $D(r)$, $L(s)$ are defined as in Relation(3.1). Recall that the radius of the essential spectrum of a bounded linear operator B is denoted by $r_e(B)$ and the radius of the spectrum of B is denoted by $\gamma(B)$. As remarked earlier,

$$r_e(T_{D(r),L(s)}(t)) = r_e(T_{D_0(r)}(t)) = \gamma(T_{D_0(r)}(t)) = e^{a_{D_0(r)}t}$$

for all $t \geqslant 0$. If $D_0(r)$ is s.l.d. at r, then Theorem 4.3 implies this is equivalent to $\sup\{a_{D_0(r)}: r \in (R^+)^N\} < -\alpha < 0$. Therefore, for any $0 < a < \alpha$, there is an interval $I_r \subseteq (R^+)^N$ countaining r and an interval $I_s \subseteq (R^+)^M$ containing s such that only a fixed finite number p of eigenvalues of $T_{D(\bar{r}),L(\bar{s})}(t)$ lie outside the circle e^{-at} for $(\bar{r}, \bar{s}) \in I_r \times I_s$. These eigenvalues must be of the form $e^{\lambda_j(r,s)t}$ where $\lambda_j(r,s) \in \sigma(A_{D(\bar{r}),L(\bar{s})})$, $j = 1,2,\ldots, p$. If it is assumed that the NFDE(D(r), L(s)) is u.a.s., then $\mathrm{Re}\lambda_j(r,s) < 0$ for $j=1,2,\ldots, p$. Rouché's theorem implies that one can further restrict the intervals I_r, I_s in such a way that $\mathrm{Re}\lambda_j(\bar{r},\bar{s}) < 0$ for all $(\bar{r},\bar{s}) \in I_r \times I_s$. Thus, the NFDE$(D(\bar{r}),L(\bar{s}))$ is u.a.s. for all $(\bar{r},\bar{s}) \in I_r \times I_s$ and the NFDE(D(r),L(s)) is s.l.d. at (r, s). Using this fact and Lemma 3.1, we have

<u>Theorem 5. 1.</u> The NFDE(D(r), L(s)) is s.l.d. at (r, s) if and only if $D_0(r)$ is s.l.d. at r.

<u>Conjecture.</u> Consider the NFDE(D(r), f(s)) where $D(r)$ is the same linear operator as before and

$$f(s)\phi = g(\phi(-s_1),\ldots, \phi(-s_n), \int_{-s_0}^{0} B(\theta)\phi(\theta)d\theta)$$

and g is nonlinear. Then the NFDE(D(r), f(s)) is s.l.d. at

(r, s) if and only if $D_0(r)$ is s.l.d. at r.

§6. Stability globally in the delays for NFDE(D(r), L(s)).

The following problem is discussed in this section.

Problem. Find necessary and sufficient conditions on the coefficients in $(D_0(r), L_0(s))$ defined in Relation(3.1) in order to have u.a.s. of the NFDE$(D_0(r), L_0(s))$ for all values of r, s.

Let us first consicer the scalar equation. One can show that there can never be u.a.s. for all r, s unless $L_0(s)$ has the form,

$$(6.1) \qquad L_0(s)\phi = B\phi(0) - \sum_{k=1}^{M} B_k\phi(-s_k), \; B \neq 0.$$

Therefore, we consider $D_0(r)$ as in (3.1) and $L_0(s)$ as in (6.1). If we write

$$(6.2) \qquad L_0(s)\phi = -BD_1(s)\phi$$

then the following result is true.

Theorem 6.1[8]. The scalar NFDE$(D_0(r), L_0(s))$ with $D_0(r)$ in (3.1) and $L_0(s)$ in (6.2) is stable for all values of r, s if and only if $B > 0$ and the difference operators $D_0(r)$, $D_1(s)$

are u.a.s. for r, s.

we prove a more general theorem below. First, we try to make some extensions to the matrix case. For the matrix case, one can prove the following sufficiency condition for stability.

Theorem 6. 2. For the vector $NFDE(D_0(r), L_0(s))$, suppose $D_0(r)$ is given as in (3.1) and $L_0(s)$ as in (6.1), (6.2) with B a nonsingular $n \times n$ matrix and suppose r, s are fixed nonnegative integer combinations of given numbers $(\beta_1, \ldots, \beta_q) \in (R^+)^q$, $r_k = \Sigma \alpha_{k_j} \beta_j$, $s_k = \Sigma \gamma_{k_j} \beta_j$ for nonnegative integers α_{k_j}, γ_{k_j}. Then we may write $D_0(r) = D_0(\beta)$, $D_1(s) = D_1(\beta)$. If $D_0(\beta)$, $D_1(\beta)$ are s.l.d. at β and $\text{Re}\lambda B < 0$, then the $NFDE(D_0(\beta), L_0(\beta))$ is stable for all $\beta \in (R^+)^q$.

Proof: The characteristic equation is given as

$$\Delta(\beta, \lambda) \overset{\text{def}}{=} \det[\lambda D_0(\beta)e^{\lambda\cdot} - BD_1(\beta)e^{\lambda\cdot}] = 0.$$

We shall prove that there is a $\delta > 0$ such that $\text{Re}\lambda \leqslant -\delta < 0$ for all $\beta \in (R^+)^q$ if $\Delta(\beta, \lambda) = 0$. If there exists a $\beta \in (R^+)^q$ and a $\lambda(\beta)$ such that $\text{Re}\lambda(\beta) > 0$, then the fact that $\text{Re}\lambda B < 0$ and $D_0(\beta)$ is s.l.d. implies there is a $\beta \in (R^+)^q$ and a $\lambda(\beta)$ such that $\text{Re}\lambda(\beta) = 0$. Therefore, if the statement in the theorem is not true, it is enough to show that the following statement is false: There exists a $\beta \in (R^+)^q$ and a sequence $\{\lambda^j\}$ of roots of $\Delta(\beta, \lambda) = 0$ such that $\text{Re}\lambda^j \longrightarrow 0$ as $j \longrightarrow \infty$. If this statement is true, then the almost periodicity of $D_0(\beta)e^{\lambda\cdot}$, $D_1(\beta)e^{\lambda\cdot}$ in β on any line $\mathrm{Re}\lambda=$constant implies there is a sequence of roots $\lambda(\beta^j) = \alpha_j + i\omega_j$ with $\omega_j > 0$, ω_j, α_j real and $\longrightarrow 0$ as $j \longrightarrow \infty$. But this contradicts the fact that $D_1(\beta)$ is s.l.d. and the theorem is proved.

Theorem 6.2 together with Theorem 4.1 can now be used to obtain a region in the parameter space for which one has u.a.s. for all values of r, s. It should be noted that this sufficiency condition does not require that (r, s) vary independently over $(R^+)^N\times(R^+)^M$ but only that $\beta \in (R^+)^q$ is allowed to vary. For difference equations, one could always transform to a higher order system to obtain an equation involving only the independent delays β. For NFDE($D_0(r)$, $L_0(s)$), this does not seem to be possible and, therefore, it is necessary to obtain results in the form stated in Theorem 6.2.

Suppose now that the NFDE($D_0(\beta)$, $L_0(\beta)$) is u.a.s. for all $\beta \in (R^+)^q$. Then necessarily $D_0(\beta)$ is s.l.d. at β and therefore stable for all $\beta \in (R^+)^q$. Therefore, there are $\delta_1 > 0$, $\delta > 0$ such that $|\det D_0(\beta)e^{\lambda\cdot}I| \geqslant \delta_1 > 0$ if $\mathrm{Re}\lambda \geqslant -\delta$. This implies that the characteristic equation is equivalent to

$$\lambda I = - BD_1(\beta)(e^{\lambda\cdot}I)[D_0(\beta)e^{\lambda\cdot}I]^{-1}$$

Now let us suppose the equation is scalar and suppose $D_1(\beta)$ is not stable locally in the delays. Then for any $\mu < 0$, the set

$$\{D_1(\beta)e^{\mu\cdot};\ \beta \in (R^+)^q\} \supseteq \text{a neighborhood } V \text{ of } 0 \in R$$

and this neighborhood V may be chosen independent of μ. Therefore, we may always determine a β and a real solution of the characteristic equation with real part as small as desired. This contradicts the hypothesis and shows that $D_1(\beta)$ must be s.l.d. and, therefore, stable for all $\beta \in (R^+)^q$. We now show $B > 0$. Clearly $B \neq 0$. If $B < 0$, then for $\beta = 0$ we have the solution

$$\lambda = -B\,\frac{1-\Sigma C_k}{1-\Sigma A_k}\,,\quad C_k = B^{-1}B_k$$

But the fact that $D_0(\beta)$, $D_1(\beta)$ are s.l.d. implies the ratio above is > 0. Therefore, we have a solution $\lambda > 0$ which is a contradiction and we have proved the following generalization of Theorem 6.1 to the case where the delays r, s may not vary independently.

Theorem 6. 3. The scalar NFDE$(D_0(\beta), L_0(\beta))$ is u.a.s. for all $\beta \in (R^+)^q$ if and only if $D_0(\beta)$, $D_1(\beta)$ are s.l.d. and $B > 0$.

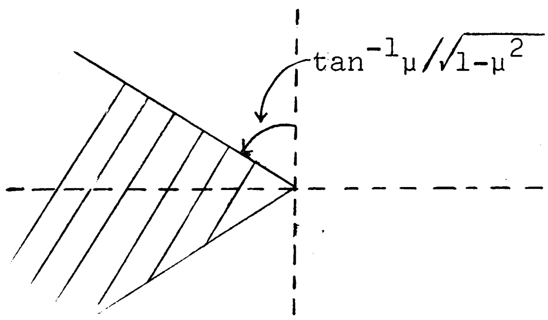
It is natural to conjecture that this result will be true for systems provided Re$\lambda B < 0$, but Infante and Tsen have suggested the following as a counterexample. Consider the equation

$$\dot{x}(t) = -B[x(t) + \mu x(t-s)] \overset{\text{def}}{=} -BD_1(s)x_t,\ -1 < \mu < 1$$

where $x \in R^n$ and B is an $n{\times}n$ matrix. Infante and Tsen assert that a necessary and sufficient condition for u.a.s. for all $s \in R$ is that

$$\mathrm{Re}[\lambda(B) \exp\,(i \tan^{-1} \mu/\sqrt{1-\mu^2})] > 0;$$

that is, $\lambda(-B)$ is in the shaded region below in the complex plane. This shows $\mathrm{Re}\lambda(-B) < 0$ is not enough.



For matrix systems with the delays r, s varying independently, Infante and Tsen make the following assertion.

<u>Theorem 6. 4.</u> The vector NFDE($D_0(r)$, $L_0(s)$) is u.a.s. for all $r \in (R^+)^N$, $s \in (R^+)^M$ if and only if

(i) $D_0(r)$ is s.l.d.

(ii) $\mathrm{Re}\lambda(B(I - \Sigma C_k) > 0$

(iii) $\det[i\omega(I - \Sigma A_j e^{i\theta_j}) - B(I - \Sigma C_k e^{i\alpha_k})] \neq 0$
for all $\omega \neq 0$ and all θ_j, α_k, where $C_k = B^{-1}B_k$.

The next important problem is to find the analogue of Theorem 6.4 for the NFDE($D_0(\beta)$, $L_0(\beta)$) with $\beta \in (R^+)^q$ defined in Theorem 6.2; that is, the case where r, s are allowed to vary only over linear subspaces $V \subseteq (R^+)^N$, $W \subseteq (R^+)^M$. As remarked before, for the difference operators, this created no essential difficulties since the dimension of the system could be increased to make the delays be exactly β. For the NFDE, this is no longer the case. Some interesting problems to discuss at the beginning would be scalar equations of the form.

$$(6.3) \quad \frac{d^n}{dt^n} D_0(r^0)x_t + \alpha_1 \frac{d^{n-1}}{dt^{n-1}} D_1(r^1)x_t + \cdots + \alpha_n D_r(r^n)x_t = 0$$

where each $D_j(r^j)$ is the usual type of difference operator and each r^j is a vector of dimension m_j. Take a solution to be one with continuous derivatives up through order $n-1$ and the n^{th} derivative of $D_0(r^0)x_t$ continuous. By transforming (6.3) to a system, the following conjecture is reasonable.

<u>Conjecture</u>: Equation (6.3) is u.a.s. for all delays if and only if $D_0(r^0)$, $D_n(r^n)$ are stable locally in delays and the roots of the equation $\lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n = 0$ have $\mathrm{Re}\lambda < 0$.

After analyzing system (6.3) in the spirit above, one should able to make some connections with the singular perturbation problems considered in [7], [9].

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